ON A CERTAIN CLASS OF INFINITELY DIVISIBLE DISTRIBUTIONS

BY

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ABSTRACT

We characterize the class of distribution functions $\Phi(x)$, which are limits in the following sense: there exist a sequence of independent and equally distributed random variables $\{\xi_n\}$, numerical sequences $\{a_k\}$, $\{b_k\}$ and natural numbers $\{n_k\}$ such that

$$\lim_{k \to \infty} \text{Prob} \left\{ \frac{1}{a_k} \sum_{k=1}^{n_k} \xi_k - b_k < x \right\} = \Phi(x)$$

and

$$\liminf_{k \to \infty} (n_k / n_{k+1}) > 0.$$

1. Introduction

Feller ([2]: 538) considers Lévy's example of a characteristic function (cf)

$$\phi(t) = \exp \sum_{k=-\infty}^{+\infty} 2^{-k+1} (\cos (2^k t) - 1), \tag{1.1}$$

which, while it is not stable, nevertheless has the curious property that

$$\phi^2(t) \equiv \phi(2t). \tag{1.2}$$

We will deal here with distribution functions (df), whose cf's satisfy a somewhat more general identity than (1.2). It is the purpose of this paper to investigate the role of such df's as limit distributions for normed sums of independent, identically distributed random variables.

It is well known that two classes of df's play a central role in the theory of limit distributions for such sums: the class of infinitely divisible (i.d.) laws and the class of stable laws. The formal definition and a detailed study of the above classes are

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given in [3]. We will just recall two problems, in connection with which these laws appear as limit distributions.

Let $\{\xi_n\}$ be a sequence of independent random variables having the same df $F(x)$; then the df of the sum $s_n = \sum_{k=1}^{n} \xi_k$ will be denoted by $F_n^*(x)$. Thus, the df of the normed sum $(s_n/a_n - b_n)$, where $a_n > 0$ and b_n are real numbers, is

$$F_n^*(a_n x + a_n b_n). \tag{1.3}$$

Lévy [6], [3] proved that a df $\Phi(x)$ can be the limit of a sequence of the form (1.3) if and only if it is stable. This problem was generalized by Khintchine. He showed [4], [3] that the totality of all partial limits of sequences of the form (1.3) coincides with the class of i.d. laws. Thus, for every i.d. distribution $\Phi(x)$ there exist a df $F(x)$, numerical sequences $\{a_k\}$ and $\{b_k\}$, and an increasing sequence of natural numbers $\{n_k\}$ such that

$$\lim_{k\to\infty} F_{n_k}^*(a_k x + a_k b_k) = \Phi(x) \tag{1.4}$$

at every continuity point of the limit distribution $\Phi(x)$.

The sequence $\{n_k\}$ is in general "sparse": the method of proof of Khintchine's result yields sequences $\{n_k\}$ which increase so rapidly that

$$\lim_{k\to\infty} n_k \sum_{i=k+1}^{\infty} (c_i/n_i) = 0,$$

where $\{c_i\}$ is a non-decreasing sequence.

It is natural to ask by how much we cut down the totality of partial limits of sequences (1.3) if we restrict ourselves to sequences $\{n_k\}$ that are "dense" in some sense. For example, we can require the existence of the limit

$$\lim_{k\to\infty} (n_k/n_{k+1}) = r, \tag{1.5}$$

or, more generally, that

$$\liminf_{k\to\infty} (n_k/n_{k+1}) = r, \tag{1.6}$$

where $r > 0$.

The df $\Phi(x)$ will be called a *partial limit of rank r*, or *r-limit*, if there exist $\{F(x), a_k, b_k, n_k\}$ such that the relations (1.4) and (1.6) hold.

In this case $F(x)$ is said to be *partially attracted with rank r to the* df $\Phi(x)$, or $F(x)$ is said to belong to the *r-attraction domain of* $\Phi(x)$.

Let C_r denote the set of all r-limit df's. Our aim is to characterize the class

$$C = \bigcup_{0<\cdot\leq 1} C_r.$$

In particular, we shall show that if we replace assumption (1.6) by the stronger condition (1.5), then neither the class of r-limit distributions nor their domain of partial attraction is diminished.

It should be noted that an analogous problem was considered by us in connection with limit distributions for the maximal term of a variational series and we use here concepts and arguments that were employed in the papers [7], [8].

2. Characterization of the class C_r

The cf's of the distributions $F(x)$ and $\Phi(x)$ will be denoted always by $f(t)$ and $\phi(t)$, respectively.

LEMMA 2.1. *Let $\Phi(x)$ be a proper* df *satisfying* (1.4). *If the sequence of the ratios $\{n_k/n_{k+1}\}$ has a partial limit r, where $0 < r < 1$, i.e., $\{n_{k(s)}\}$ is a subsequence such that*

$$\lim_{s\to\infty} (n_{k(s)}/n_{k(s)+1}) = r, \tag{2.1}$$

then the following finite limits exist:

$$\begin{aligned} &\lim_{s\to\infty} (a_{k(s)}/a_{k(s)+1}) = a, \\ &\lim_{s\to\infty} (b_{k(s)}a_{k(s)}/a_{k(s)+1} - b_{k(s)+1}n_{k(s)}/n_{k(s)+1}) = b, \end{aligned} \tag{2.2}$$

where

$$0 < a < 1, \tag{2.3}$$

and we have the identity

$$\phi(at)\exp(ibt) \equiv \phi^r(t). \tag{2.4}$$

PROOF. Condition (1.4) may be written in the form

$$\lim_{k\to\infty} f^{n_k}(t/a_k)\exp(-itb_k) = \phi(t). \tag{2.5}$$

Hence, because of (2.1) we have also

$$\lim_{s\to\infty} f^{n_{k(s)}}(t/a_{k(s)+1})\exp(-itb_{k(s)+1}n_{k(s)}/n_{k(s)+1}) = \phi^r(t).$$

From what we said in the Introduction about Khintchine's result, it is obvious, that $\phi(t)$ is an i.d. cf. Therefore $\phi^r(t)$ is the cf of some proper df, which will be denoted by $\Phi_r(x)$. Thus the last equality can be rewritten as

$$\lim_{s\to\infty} F^*_{n_{k(s)}}(a_{k(s)+1}x + a_{k(s)+1}b_{k(s)+1}n_{k(s)}/n_{k(s)+1}) = \Phi_r(x).$$

But, by another theorem of Khintchine [5], [3], a sequence of types can converge to at most one proper type. Hence, juxtaposing the last equation with (1.4), we conclude immediately that the df's $\Phi(x)$ and $\Phi_r(x)$ must belong to the same type. In other words, there exist constants $a > 0$ and b such that

$$\Phi\left(\frac{x-b}{a}\right) \equiv \Phi_r(x).$$

This proves (2.4). It is easy to see that these constants are the limits (2.2). Since $\phi(t)$ is i.d., $\phi(t) \neq 0$. Supposing that $a = 1$, we would have by (2.4) that $|\phi(t)|^{1-r} \equiv 1$, which is impossible, since $0 < r < 1$ and $\Phi(x)$ is a proper df. Since $|\phi(t)| \leqq 1$, again by (2.4) we would have $|\phi(at)| \geqq |\phi(t)|$, and hence, for every t and natural n, also $|\phi(t)| \geqq |\phi(t/a^n)|$. Therefore, supposing $a > 1$ and passing to the limit for $n \to \infty$, we would have $|\phi(t)| \geqq \phi(0)| = 1$ for every t. But this is impossible, since $\Phi(x)$ is a proper df. Thus inequalities (2.3) are proved.

LEMMA 2.2. *Let* (2.5) *hold for some* $\{n_k, a_k, b_k\}$, *where* $n_k(n_{k+1} \geqq n_k + 1)$ *are positive numbers, not necessarily integers. Then there exist* b'_k *such that* (2.5) *holds for* $\{[n_k], a_k, b_k\}$ ($[n_k]$ *is the integral part of* n_k).

PROOF. Indeed, on raising both sides of (2.5) to the power $[n_k]/n_k$ one obtains the required result with $b'_k = b_k[n_k]/n_k$.

A characteristic property of the class $C_r(0 < r < 1)$ is given by the following

THEOREM 2.1. *A* df $\Phi(x)$ *belongs to the class* $C_r(0 < r < 1)$ *if and only if its* cf $\phi(t)$ *has the following property*:

There exist constants $a = a(r)$ *and* $b = b(r)$ *such that the identity* (2.4) *holds.*

PROOF. Since each improper df belongs to each C_r and satisfies condition (2.4), we may assume that $\Phi(x)$ is a proper df.

The necessity of (2.4) is proved by Lemma 2.1. Let the cf $\phi(t)$ satisfy condition (2.4). By iterating this relation n times we get the identity

$$\phi(a^n t)\exp\left(itb\sum_{s=0}^{n-1} a^s r^{n-s-1}\right) \equiv \phi^{r^n}(t), \tag{2.6}$$

which is valid for every natural n. Hence, $\phi^{r^n}(t)$ is a cf for every n. Since $0 < r < 1$, each positive number λ can be represented in the form

$$\lambda = \sum_{n=1}^{\infty} k_n r^n,$$

where k_n are non-negative integers. On the other hand, the product of cf's and the limit of a sequence of cf's are cf's. Thus we conclude that if a cf $\phi(t)$ has the property (2.4), then $\phi^{\lambda}(t)$ is a cf for each $\lambda > 0$ and, therefore, $\phi(t)$ is i.d.

Now, let $\{\xi_n\}$ be a sequence of mutually independent random variables having the same df $F(x) = \Phi(x)$. Let us denote

$$\nu_k = r^{-k},\ a_k = a^{-k},\ \beta_k = -\frac{b}{r}\sum_{s=0}^{k-1}(a/r)^s, \qquad k = 1,\cdots.$$

Then by (2.6) we get

$$f^{\nu_k}(t/a_k)\exp(-it\beta_k) = \phi(t),$$

since $f(t) = \phi(t)$.

It follows from Lemma 2.2 that the sequence $\{\beta_k\}$ can be replaced by an appropriate sequence $\{b_k\}$ in such a way that we get (2.5) with

$$n_k = [\nu_k] = [r^{-k}].$$

Since this $\{n_k\}$ satisfies condition (1.5), the sufficiency of (2.4) is proved.

Let us note the following propositions that follows immediately from the above arguments:

COROLLARY 2.1. *Replacing assumption* (1.6) *by* (1.5) *does not change the class* C_r.

COROLLARY 2.2. *Each* df *from* C_r *is attracted by itself with rank* r.

It is natural to denote the class of all i.d. df's by C_0. For each i.d. df $\Phi(x)$ let us denote by $R(\Phi)$ the set of ranks r $(0 \leqq r \leqq 1)$ for which $\Phi(x)$ is an r-limit. By our assumption about $\Phi(x)$ the set $R(\Phi)$ is not empty. It is easy to see that this set has a maximum, which will be called the *maximal rank* of partial attraction of the df $\Phi(x)$ and denoted by r_0:

$$r_0 = r_0(\Phi) = \max_r R(\Phi).$$

It can be shown (cf. [7], [8]) that the set $R(\Phi)$ is uniquely determined by its maximum r_0:

THEOREM 2.2. a) $0 < r_0 < 1$ *if and only if*

$$R(\Phi) = \{r : r = r_0^m;\ m = 1,\cdots,\infty\}.$$

b) $r_0 = 1$ *if and only if* $R(\Phi) = [0,1]$.

If $R(\Phi)$ contains two numbers such that the ratio of their logarithms is irrational, then $r_0 = 1$.

It follows from the definition of C_r that $C_1 = \bigcap_{0 \leqq r \leqq 1} C_r$. By Theorem 2.1 it can be shown that C_1 coincides with the class of stable laws. However, we can also prove this fact bypassing the formal definition of the stable laws, and proving a somewhat stronger proposition:

THEOREM 2.3. *If a* df $F(x)$ *is partially attracted to the* df $\Phi(x)$ *with the rank* $r = 1$, *then* $F(x)$ *belongs to the (full) domian of attraction of* $\Phi(x)$.

PROOF. Let us suppose that (2.5) holds, where

$$\lim_{k \to \infty} (n_k / n_{k+1}) = 1.$$

Then after some simple transformations we conclude that for every t

$$\lim_{k \to \infty} f^{n_{k+1} - n_k} (t / a_k) \exp(- itb_k(n_{k+1} / n_k - 1)) = 1.$$

Since $|f(t)| \leqq 1$, we get also

$$\lim_{k \to \infty} |f^s(t / a_k)| = 1$$

uniformly in s, $0 \leqq s \leqq n_{k+1} - n_k$. Therefore, ([3, Th. 3, p. 57]) there exist numbers β_{ks} $(k = 1, \cdots, ; 0 \leqq s \leqq n_{k+1} - n_k)$ such that for each t

$$\lim_{k \to \infty} f^s(t / a_k) \exp(- it\beta_{ks}) = 1.$$

On the other hand, every natural number n $(n \geqq n_1)$ can be represented in the form $n = n_k + s(k,n)$, where $0 \leqq s = s(k,n) < n_{k+1} - n_k$.

Let us supplement the initial sequences $\{a_n\}$ and $\{b_n\}$ by putting

$$a_n = a_k, \; b_n = b_k + \beta_{ks} \text{ for } n_k \leqq n \leqq n_{k+1}.$$

Then it is easy to see that

$$\lim_{n \to \infty} f^n(t / a_n) \exp(- itb_n) = \phi(t).$$

We conclude with a property of the class C.

THEOREM 2.4. *The class C does not contain any lattice distribution.*

PROOF. It suffices to prove that if $\phi(t)$ is a cf of a proper df from class C, then $|\phi(t)| < 1$ for $t \neq 0$. Indeed, suppose that for some $t_0 \neq 0$ we have $|\phi(t_0)| = 1$, where $\phi(t)$ satisfies the condition (2.4). By (2.6) we would have also

$$|\phi(a^n t_0)| = |\phi(t_0)|^{r^n} = 1$$

for every natural n. Denoting $t_n = a^n t_0$, $n = 1, \cdots$, we would have $\lim_{n\to\infty} t_n = 0$, since $0 < a < 1$, and also $|\phi(t_n)| = 1$ for every n. But this is inconsistent with the assumption that $\phi(t)$ is the cf of a proper df. ([3, Th. 2, p. 56]).

3. Remarks about the domain of partial attraction of distributions from the class C

A sequence $\{n_k\}$ which satisfies condition (1.6) may contain no subsequence $\{n_{k(s)}\}$ such that

$$\lim_{s\to\infty} (n_{k(s)}/n_{k(s+1)}) > 0.$$

Nevertheless, it turns out that conditions (1.5) and (1.6) are also equivalent in the definition of the domain of partial attraction (with positive rank) of distributions from C. Moreover, the following theorem holds:

THEOREM 3.1. *If $F(x)$ and $\Phi(x)$ are* df's *such that for some r ($0< r < 1$) and $\{n_k, a_k, b_k\}$ relations* (1.4) *and* (1.6) *hold, then for every r in $R(\Phi)$ there exist $\{n_k, a_k, b_k\}$ such that* (1.4) *and* (1.5) *hold.*

PROOF. Let r_0 be the maximal rank of partial attraction of the df $\Phi(x)$, which is assumed to be proper. By Theorem 2.2 it is enough to show that under our conditions there exist $\{\bar{n}_k, \bar{a}_k, \bar{b}_k\}$ such that for every t

$$\lim_{k\to\infty} f^{\bar{n}_k}(t/\bar{a}_k)\exp(-it\bar{b}_k) = \phi(t), \tag{3.1}$$

where

$$\lim_{k\to\infty} (\bar{n}_k/\bar{n}_{k+1}) = r_0. \tag{3.2}$$

Let us consider two cases.

CASE I. $0 < r_0 < 1$. By Theorem 2.1, a_0 ($0 < a_0 < 1$) and b_0 exist such that

$$\phi(a_0 t)\exp(ib_0 t) \equiv \phi^{r_0}(t). \tag{3.3}$$

Let s be an arbitrary natural number. Denote

$$n_{ks} = n_k r_0^{-s}, \quad a_{ks} = a_k a_0^{-s}, \tag{3.4}$$

$$b_{ks} = b_k a_0^s / r_0^s - (b_0/r_0) \sum_{j=0}^{s-1} (a_0/r_0)^j.$$

Then, because of (3.3) and (2.6) it is easy to see, that if we have (1.4) or, what is the same, (2.5), then also

$$\lim_{k\to\infty} f^{n_{ks}}(t/a_{ks})\exp(-itb_{ks}) = \phi(t). \tag{3.5}$$

By Lemma 2.1, the df $\Phi(x)$ is r-limit for every r ($0 < r < 1$), which is a partial limit of the sequence $\{n_k/n_{k+1}\}$. On the other hand, by Theorem 2.2 (a), there exists a natural number m such that for the value r appearing in (1.6) we have $r = r_0^m$. Therefore, the set of partial limits of the sequence $\{n_k/n_{k+1}\}$ does not include any other number except

$$1, r_0, r_0^2, \cdots, r_0^m = r.$$

It follows from here (cf. [8]*) that there exists a subsequence $\{n_{k(s)}\}$ such that

$$r = r_0^m = \liminf_{s\to\infty} (n_{k(s)}/n_{k(s+1)}) \leqq \limsup_{s\to\infty} (n_{k(s)}/n_{k(s+1)}) \leqq r_0.$$

Therefore, in order to simplify notation we may assume that for every k the original sequence $\{n_k\}$ satisfies the inequalities

$$r_0^{m+\frac{1}{2}} < (n_k/n_{k+1}) < r_0^{\frac{1}{2}}. \tag{3.6}$$

Consider the intervals

$$\Delta_p = [r_0^{p+\frac{1}{2}}, r_0^{p-\frac{1}{2}}), \ p = 1, \cdots, m.$$

By (3.6) we have for every k an integer $p = p(k)$ such that

$$(n_k/n_{k+1}) \in \Delta_p.$$

Let us supplement the sequence $\{n_k\}$ in the following way: if $p(k) \geqq 2$, then we insert between n_k and $n_{k+1}(p-1)$ numbers n_{ks} of the form (3.4), where $s = 1, \cdots, p-1$. If we put $n_k = n_{k0}$, $n_{k+1} = n_{kp}$, then, in view of (3.6) and the definition of n_{ks}, we conclude that for every k and s ($0 \leqq s \leqq p-1$) we get

$$r_0^{\frac{3}{2}} < (n_{k,s+1}) < r_0^{\frac{1}{2}}.$$

Thus we obtain an increasing sequence, which will be denoted by $\{\bar{n}_k\}$. We have obviously

$$r_0^{\frac{3}{2}} < (\bar{n}_k/\bar{n}_{k+1}) < r_0^{\frac{1}{2}}. \tag{3.7}$$

Analogously we supplement the sequences $\{a_k\}$ and $\{b_k\}$ according to (3.4) and denote the supplemented sequences by $\{\bar{a}_k\}$ and $\{\bar{b}_k\}$. Since (3.5) is valid for every s, we will have (3.1) for our $\{\bar{n}_k, \bar{a}_k, \bar{b}_k\}$. Since we are considering the case $0 < r_0 < 1$, by Theorem 2.2(a) and inequalities (3.7) we conclude that (3.2) holds. By Lemma 2.2

* I take the opportunity of correcting a misprint: inequalities (3.8) on p. 211 [8] must be written as $r_0^{\frac{1}{2}} < q < 1$.

we can modify the definition of the sequences $\{\bar{n}_k\}$ and $\{\bar{b}_k\}$ preserving the equations (3.1) and (3.2), so that the n_k will be natural numbers.

CASE II. $r_0 = 1$. In this case the df $\Phi(x)$ is stable and the canonical representation of its cf $\phi(t)$ has the form

$$\log \phi(t) = i\gamma t - c|t|^{\alpha}\left(1 + i\beta \frac{t}{|t|}\omega(t,\alpha)\right), \tag{3.8}$$

where α, β, γ, c are constants ($0 < \alpha \leqq 2$, $|\beta| \leqq 1$, $c \geqq 0$) and

$$\omega(t,\alpha) = \begin{cases} tg\,\dfrac{\pi}{2}\alpha, & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi}\log|t|, & \text{if } \alpha = 1. \end{cases}$$

Now, for every r ($0 < r < 1$) there exist a and b such that we have the identity (2.4). Using (3.8) one checks that the constants a, b and r are related by the equations

$$r = a^{\alpha},\ b = b(r,a) = \begin{cases} \gamma(r-a), & \text{if } \alpha \neq 1, \\ \dfrac{2}{\pi}c\beta a \log a, & \text{if } \alpha = 1. \end{cases} \tag{3.9}$$

It follows from here that if we have (2.4) for some r, a and b, then for each number $d \geqq 0$ we also have the identity

$$\phi(a^d t)\exp(itb(r^d, a^d)) \equiv \phi^{r^d}(t)$$

(which can be regarded as a continuous analogue of (2.6)).

It can be shown (cf. [8]) that, in view of (1.6), for every q ($0 < q < 1$) there exists a subsequence $\{n'_k\}$ such that $qr \leqq (n'_k/n'_{k+1}) < q$. Therefore, without loss of generality, we can assume that

$$q_1 < (n_k/n_{k+1}) < q_2, \tag{3.11}$$

where $0 < q_1 < q_2 < 1$.

Let us supplement the sequence $\{n_k\}$ by inserting between n_k and n_{k+1} numbers n_{km}, defined by

$$n_{km} = n_k(n_{k+1}/n_k)^{m/k};\ m = 1,\cdots,k-1. \tag{3.12}$$

By (3.11) the series $\sum_{k=1}^{\infty} n_k^{-1}$ converges and, therefore, $k/n_k \to 0$ and also

$k/(n_{k+1} - n_k) \to 0$ for $k \to \infty$. Hence, for sufficiently large k, we will have $[n_{k,m+1}] > [n_{km}]$. The sequence, supplemented in this way, will be denoted by $\{\bar{n}_k\}$. By (3.11) we have

$$q_1^{1/k} < (n_k/n_{k+1})^{1/k} < q_2^{1/k}.$$

Hence we conclude that

$$\lim_{k\to\infty} (n_k/n_{k+1}) = 1,$$

which coincides — in the present case — with (3.2).

Let us supplement the sequences $\{a_k\}$ and $\{b_k\}$ by putting

$$a_{km} = a_k(a_{k+1}/a_k)^{m/k},$$

$$b_{km} = \begin{cases} (b_k + \gamma)(n_{k+1}a_k/n_k a_{k+1})^{m/k} - \gamma, & \text{if } \alpha \neq 1, \\ b_k - \dfrac{2}{\pi}\dfrac{m}{k} c\beta \log(a_k/a_{k+1}), & \text{if } \alpha = 1, \end{cases} \tag{3.13}$$

$$m = 1, \cdots, k-1.$$

The supplemented sequences will be denoted by $\{\bar{a}_k\}$ and $\{\bar{b}_k\}$.

In order to prove (3.1) we will show that each subsequence of the sequence

$$\{f^{\bar{n}_k}(t/\bar{a}_k)\exp(-it\bar{b}_k)\}$$

contains a subsequence which converges to $\phi(t)$. By (3.12), each subsequence $\{\bar{n}_{k(s)}\}$ can be represented in the form

$$\bar{n}_{k(s)} = n_{k(s)}(n_{k(s)+1}/n_{k(s)})^{m(s)/k(s)}.$$

Since $1 \leqq m \leqq k-1$, we can assume the existence of the limit

$$\lim_{s\to\infty} (m(s)/k(s)) = d, \tag{3.15}$$

where $0 \leqq d \leqq 1$, and the limit (2.1), where $0 < r < 1$ by (3.11). By Lemma 2.1, for this subsequence $\{\bar{n}_{k(s)}\}$ the limits (2.2) also exist, and the values of a, b and r must satisfy the equations (3.9), since $\phi(t)$ is now the cf of a stable law. In addition we have the identity (2.4). It is easy to see from (3.13), (3.14) and (3.15) that the subsequences $\{n_{k(s)}, a_{k(s)}, b_{k(s)}\}$ and $\{\bar{n}_{k(s)}, \bar{a}_{k(s)}, \bar{b}_{k(s)}\}$ are related by the following asymptotic equations for $s \to \infty$:

$$\begin{aligned} &n_{k(s)}/(r^d \bar{n}_{k(s)}) \to 1, \quad a_{k(s)}/(a^d \bar{a}_{k(s)}) \to 1, \\ &a^d b_{k(s)} - r^d \bar{b}_{k(s)} \to b(r^d, a^d), \end{aligned} \tag{3.16}$$

where $b(r, a)$ is given by (3.9).

On the other hand, since by assumption we have (1.4), obviously

$$\lim_{s\to\infty} f^{n_{k(s)}}(t/a_{k(s)})\exp(-itb_{k(s)}) = \phi(t).$$

Hence, in view of (3.16), we get

$$\lim_{s\to\infty} f^{\bar{n}_{k(s)}}(t/\bar{a}_{k(s)})\exp(-it\bar{b}_{k(s)})$$

$$= \phi^{r^{-d}}(a^d t)\exp(itb(r^d, a^d)/r^d).$$

But for our a, b and r the identity (2.4) holds. Therefore, for each non-negative number d and, in particular, for the d which appears in (3.15), the identity (3.10) necessarily holds. This, in turn, means that the right-hand side of the last equation coincides with $\phi(t)$. Thus (3.1) is proved. As we saw in the previous case, the passage to natural numbers $\bar{n}_k$ is accomplished with the help of Lemma 2.2, which completes the proof.

Let us note the obvious proposition:

COROLLARY 3.1. *The domain of partial attraction with positive rank of a stable law coincides with its (full) domain of attraction.*

Let $\Phi(x)$ be an i.d. df, $\phi(t)$ its cf, λ a positive number. The df whose cf is $\phi^{\lambda}(t)$ will be denoted by $\Phi_{\lambda}(x)$. The notion of distribution type, which is essential in the theory of limit distributions for sums of independent random variables, will now be extended in the class of i.d. df's:

Two i.d. df's $\Phi(x)$ and $\Psi(x)$ are said to belong to the same distribution *"family"* if there exist constants $a > 0$, b and $\lambda > 0$ such that

$$\Psi(x) \equiv \Phi_{\lambda}(ax + b)$$

(at every continuity point of the both sides of the equation).

The following propositions are easily inferred from the proof of Theorem 2.1:

COROLLARY 3.2. (a) *If the* df $\Phi(x)$ *belongs to the class* C_r, *then the entire family* Φ *belongs to* C_r.

(b) *If an* i.d. df $F(x)$ *belongs to the r-attraction domain of the* df $\Phi(x)$, *then the entire family F is contained in the r-attraction domain of each* df *belonging to the family* Φ.

It is easy to prove the following proposition (cf. [8, Th. 4.1]):

THEOREM 3.2. *Let* $\Phi(x)$ *and* $\Psi(x)$ *be two proper* i.d. df's *and let* $F(x)$ *be partially attracted by both of them. If* $F(x)$ *is partially attracted by* $\Phi(x)$ *or* $\Psi(x)$ *with a positive rank, then* $\Phi(x)$ *and* $\Psi(x)$ *belong to the same family.*

Theorems 2.2 and 3.1 show that the numerical value of the limits (1.5) or (1.6) is not essential and the value r was introduced in order to facilitate the formulation of corresponding propositions. Indeed, for a given i.d. df $\Phi(x)$ it is meaningful to speak of only two kinds of domains of partial attraction: those with "*zero density*" and those with "*positive density*". Let us denote by $D_p(\Phi)$ the entire domain of partial attraction of the df $\Phi(x)$, and by $D_p^+(\Phi)$ those of positive density. Then, it is easy to prove by the previous theorem the following.

THEOREM 3.3. *For each proper* i.d. df $\Phi(x)$ *we have*

$$D_p(\Phi) - D_p^+(\Phi) \neq \varnothing.$$

PROOF. It follows from Doeblin's [1] results that there exist "universal" df's that are partially attracted by every i.d. df. Let $F(x)$ be such a universal df and $\Psi(x)$ an i.d. df which does not belong to the class C so that $D_p^+(\Psi) = \varnothing$. (For $\Psi(x)$ we can take the Poisson distribution, since as a lattice distribution it does not belong to the class C by Theorem 2.4.) Thus we have

$$F(x) \in D_p(\Phi) \text{ and } F(x) \in D_p(\Psi).$$

It is easy to see that $F(x) \notin D_p^+(\Phi)$. Indeed, assuming the contrary, we would have by Theorem 3.2 and Corollary 3.2 that $F(x) \in D_p^+(\Psi)$ too, which contradicts the choice of $\Psi(x)$. Thus our proposition is proved.

4. The canonical representation

We now give a constructive characterization of the class C.

THEOREM 4.1. (I) *In order that the function $\phi(t)$ be the* cf of a df $\Phi(x)$ *of class C, it is necessary and sufficient that its logarithm be representable in the form*

$$\log \phi(t) = i\gamma t - \frac{\sigma^2 t^2}{2} + \int_{-\infty}^{0}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dH_1(u) + \int_{0}^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dH_2(u), \tag{4.1}$$

where γ and σ^2 real constants, and

$$H_1(u) = h_1(\log|u|)/|u|^{\alpha}, \qquad (u < 0),$$
$$H_2(u) = -\,h_2(\log u)/u^{\alpha}, \qquad (u > 0), \tag{4.2}$$

where α is a constant,

$$(4.3) \qquad 0 < \alpha < 2,$$

and $h_1(x)$ *and* $h_2(x)$ *are bounded, non-negative functions defined in* $(-\infty, \infty)$, *such that*

(I_1) *the ratio* $h_k(x)/e^{\alpha x}$ *is non-increasing in*

$$(4.4) \qquad (-\infty, +\infty), \ (k = 1, 2);$$

(I_2) *there exists a positive constant* T *such that*

$$(4.5) \qquad h_k(x + T) - h_k(x) \equiv 0, \qquad (k = 1, 2)$$

(*at every continuity point of the functions* $h_k(x)$ *and* $h_k(x + T)$).

In addition, if $\sigma^2 > 0$, *then*

$$(4.6) \qquad h_1(x) = h_2(x) \equiv 0.$$

(II) *The representation of the* cf *of a distribution from class C in the above form is unique in the following sense:*

(II_1) *The constants* γ *and* σ^2 *are uniquely determined.*

(II_2) *The functions* h_1 *and* h_2 *are uniquely determined if we identify any two functions of bounded variation in every finite segment that coincide at their continuity points.*

(II_3) *The constant* α *is uniquely determined if* $\phi(t)$ *is the* cf *of a proper, non-gaussian* df (*in the exceptional cases the value of* α *is not essential*).

(III) *If* T_0 *is the minimal common period of the functions* h_1 *and* h_2, *then the maximal rank* r_0 *of partial attraction of the* df $\Phi(x)$ *is given by*

$$(4.7) \qquad r_0 = \exp(-\alpha T_0).$$

In particular, if both the functions h_1 *and* h_2 *are constants, then* $\Phi(x)$ *is a stable law and* $r_0 = 1$.

PROOF. (*I*). *Necessity.* Let $\phi(t)$ be the cf of a df from class C. Thus, for some r, a and b $(0 < r, a < 1)$ we have the identity (2.4). which we will write now in the form

$$(4.8) \qquad \log \phi(at) + ibt \equiv r \log \phi(t).$$

It is well known that the canonical representation of an i.d. cf is given by (4.1), where γ and σ^2 are constants, and the functions $H_1(u)$ and $H_2(u)$ are non-decreasing in the intervals $(-\infty, 0)$, $(0, +\infty)$ respectively, and satisfy the relations

$$H_1(-\infty) = H_2(+\infty) = 0 \tag{4.9}$$

and

$$\int_{-\varepsilon}^{0} u^2 dH_1(u) + \int_{0}^{\varepsilon} u^2 dH_2(u) < \infty \tag{4.10}$$

for every finite $\varepsilon > 0$.

This representation is unique if we identify any two non-decreasing functions that coincide at all of their continuity points.

It can be shown by simple manipulations that if $\log\phi(t)$ is of the form (4.1) and a is a positive number, then $\log\phi(at)$ can be represented in the form

$$\begin{aligned} \log\phi(at) &= it(a\gamma + \gamma_1 + \gamma_2) - \frac{\sigma^2 a^2 t^2}{2} \\ &+ \int_{-\infty}^{0}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dH_1(u/a) + \int^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)dH_2(u/a), \end{aligned} \tag{4.11}$$

where

$$\gamma_k = a(1-a^2)\int \frac{z^3}{(1+a^2z^2)(1+z^2)}dH_k(z), \tag{4.12}$$

and the domain of integration is $(-\infty, 0)$ for $k = 1$ and $(0, +\infty)$ for $k = 2$. Hence, by relations (4.1), (4.8) and (4.11) and the uniqueness of the canonical representation of an i.d. cf, we conclude that the functions $H_1(u)$ and $H_2(u)$ must satisfy the identities

$$\begin{aligned} rH_1(u) &\equiv H_1(u/a), \qquad (u < 0), \\ rH_2(u) &\equiv H_2(u/a), \qquad (u > 0), \end{aligned} \tag{4.13}$$

(where u and u/a are continuity points of the functions H_1 and H_2), while the constants a, b and r are connected by the equation

$$\gamma(a - r) + b + \gamma_1 + \gamma_2 = 0, \tag{4.14}$$

γ_1, γ_2 being given by formula (4.12). In addition, if $\sigma^2 > 0$, it is also necessary that

$$r = a^2. \tag{4.15}$$

Let us consider the function $H_1(u)$. Since it is non-decreasing, it follows from (4.9) that $H_1(u) \geqq 0$. We show that either $H_1(u) \equiv 0$, or $H_1(u) > 0$ in $(-\infty, 0)$. Indeed, denote

$$\bar{u} = \sup\{u : H_1(u/a) = 0,\ u < 0\}.$$

Assuming $-\infty < \bar{u} < 0$, we would have by (4.13) that

$$H_1\left(\frac{\bar{u}}{a} - 0\right) = H_1(\bar{u} - 0) = 0,$$

which contradicts the definition of $\bar{u}$, since $\bar{u} > \bar{u}/a$.

Let us consider the case $H_1(u) > 0$ for $u < 0$. Denoting

$$\alpha = \frac{\log r}{\log a},\ T = -\log a, \tag{4.16}$$

$$h_1(x) = e^{\alpha x} H_1(-e^x),\ (-\infty < x < \infty) \tag{4.17}$$

we get by (4.13) the necessity of condition (4.5). Since the function H_1 is monotone and is finite valued in $(-\infty, 0)$, by (4.5) there exist positive numbers m and M such that

$$m \leqq h_1(x) \leqq M \tag{4.18}$$

(we can put, for example, $m = e^{\alpha x_0} H_1(-e^{x_0+T} + 0)$, $M = e^{\alpha(x_0+T)} H_1(-e^{x_0} - 0)$, where x_0 is arbitrary).

It follows from (4.17) that H_1 must be of the form (4.2), where $\alpha > 0$. Let us show that $\alpha < 2$. For every $\varepsilon > 0$ we have

$$\int_{-\varepsilon}^{0} u^2 dH_1(u) = \int_{-\varepsilon}^{0} d(u^2 H_1(u)) + \int_{-\varepsilon}^{0} -2uH_1(u)du, \tag{4.19}$$

where, under our conditions, the integrals on the right are positive. Therefore, in view of (4.10) we get

$$\left| \int_{-\varepsilon}^{0} uH_1(u)du \right| < \infty,$$

and, by (4.18), also

$$m \int_{-\varepsilon}^{0} |u|^{1-\alpha} du < \infty,$$

which proves the necessity of (4.3).

The condition (4.4) follows from the monotonicity of the function $H_1(u)$.

Quite analogously we establish the necessary form of the function $H_2(u)$, where the value of T in (4.5) for $k = 2$ must be the same as in the case $k = 1$.

The necessity of the inequalities (4.3) as obtained from (4.10) by the assumption that at least one of the functions H_1 and H_2 is not identically zero. However, by the definition (4.16) of the parameter α, the relations (4.3) and (4.15) are inconsistent. Hence it is obvious that if $\sigma^2 > 0$, then the condition (4.6) is necessary.

(II) Sufficiency of our conditions is almost evident. Condition (4.4) guarantees the monotonicity of the functions H_1 and H_2, while the boundedness of the functions h_1 and h_2 guarantees (4.9). By starting from (4.19) and the analogous equality for H_2, it is easy to show that if $0 < \alpha < 2$ and $h_k(x)$ is bounded, then the functions H_1 and H_2 satisfy inequalities (4.10) for every $\varepsilon > 0$. Thus the function $\phi(t)$, which is given by (4.1), is an i.d. cf.

Let T be a common period of h_1 and h_2. Then it is easy to verify that $\phi(t)$ satisfies the identity (4.8) if we put, for example,

$$r = \exp(-\alpha T), \ a = \exp(-T), \quad b = -\gamma(a-r) - \gamma_1 - \gamma_2.$$

REMARK 4.1. We saw above that either $h_k(x) \equiv 0$ or $h_k(x) > 0$ in $(-\infty, \infty)$. If $h_k(x) > 0$, then condition (4.4) may be written in a more convenient form

$$\log h_k(x_2) - \log h_k(x_1) \leqq \alpha(x_2 - x_1)$$

for $x_2 > x_1$.

(II) The constants γ and σ^2 and the functions H_1 and H_2 are uniquely determined (in the above-mentioned sense), since $\phi(t)$ is i.d. Therefore, it remains to observe that the function $h_k(x)$ is uniquely determined (on the set of its discontinuity points) by the function $H_k(u)$ by means of equation (4.2), and that the parameter α is uniquely determined if at least one of the functions H_1 or H_2 is not identically zero. But this is obvious.

Indeed, let us suppose, for example, that $H_1(u) \not\equiv 0$ and, therefore, $H_1(u) > 0$. Then, assuming

$$H_1(u) = h_1(\log|u|)/|u|^\alpha = h_1^*(\log|u|)/|u|^{\alpha^*}$$

and putting

$$g(x) = h_1^*(x)/h_1^*(x), \qquad \beta = \alpha^* - \alpha,$$

we obtain the identity $g(x) \equiv e^{\beta x}$, where $g(x)$ is a periodic function; but this is possible only in the case, when $\beta = 0$ and $g(x) \equiv 1$.

Let us also observe that the function $h_k(x)$ is of bounded variation in each finite segment, since it is a product of two monotonic functions.

(III) Formula (4.7) is evident. It is well known that the expression (4.1) reduces to the canonical representation of a stable law if the functions h_1 and h_2 appearing in (4.2) are constants. This fact follows also from the first part of the present theorem and Theorem 2.2(b).

REMARK 4.2. It is easy to see by (4.1), (4.2) and (4.11) that the parameter α appearing in (4.2) has the same value for all df s which belong to the same type. Therefore, it is meaningful to speak of the "*exponent*" $\alpha = \alpha(\Phi)$, which corresponds to a given type Φ from the class C.

EXAMPLE 4.1. Let us consider the cf (1.1). It is obviously the cf of a distribution from $C_{\frac{1}{2}}$. Keeping our notation, we have here $a = r = \frac{1}{2}$, $b = 0$. In the canonical representation of this cf $\gamma = \sigma^2 = 0$,

$$(4.20)\qquad \begin{aligned} &H_1(u) = 2^k \text{ if } -2^{-k+1} < u < -2^{-k}, \qquad k = 0, \pm 1, \cdots, \\ &H_2(u) = -H_1(-u), \qquad (u > 0). \end{aligned}$$

Since $a = r$, by (4.16) we have $\alpha = 1$. Hence

$$h_1(x) = 2^k e^x, \text{ if } -k \log 2 < x < -(k-1)\log 2, \qquad k = 0, \pm 1, \cdots$$

and an analogous expression holds for $h_2(x)$. Since $T_0 = \log 2$ is the minimal period of the functions h_1 and h_2, it follows from (4.7) that the maximal rank of partial attraction is $r_0 = \frac{1}{2}$.

The cf $\phi(t)$ of the above example satisfies a particular form of the identity (2.4), namely

$$(4.21)\qquad \phi(at) \equiv \phi^r(t)\,.$$

The totality of cf's that satisfy an identity of this special form is in a certain sense not closed. If a cf with property (4.21) corresponds to the random variable ξ, then the cf of the random variable $(a\xi + b)$ cannot satisfy (4.21) if $b \neq 0$. Nevertheless, it is of some interest to find these types Φ of the class C, which contain df's whose cf's satisfy the identity (4.21). An exhaustive answer to this question is given by the following proposition, which is a direct corollary of Theorem 4.1:

COROLLARY 4.1. *Every type Φ of class C with exponent $\alpha = \alpha(\Phi) \neq 1$ contains* df's *whose* cf's *satisfy an identity of the form* (4.21). *In order that the* cf $\phi(t)$ *of a* df $\Phi(x)$ *of class C satisfy* (4.21), *where* $a = r$ (i.e., $\alpha(\Phi) = 1$), *it is necessary and sufficient that*

$$(4.22)\qquad \gamma_1 + \gamma_2 = 0,$$

where γ_1 and γ_2 are defined by (4.12) *and* $a = \exp(-T)$.

For the proof it is enough to refer to (4.14).

The condition (4.22) is obviously satisfied in our Example 4.1, because we have (4.20).

Finally, let us adduce an easily provable proposition concerning symmetric distributions of class C.

THEOREM 4.2. *In order that a real* cf $\phi(t)$ *be the* cf *of a law of class C, it is necessary and sufficient that it be representable in the form*

$$\phi(t) = \exp(-|t|^{\alpha}h(\log|t|)), \tag{4.23}$$

where $h(t)$ is a continuous, periodic, strictly positive function in $(-\infty,\infty)$, α is a constant, $0 < \alpha < 2$, and $h(t) \equiv$ constant if $\alpha = 2$.

PROOF. *Necessity.* By assumption, $\phi(t)$ is a cf which satisfies (2.4). Since $\phi(t)$ is i.d., continuous and $\phi(0) = 1 > 0$, it follows that $\phi(t) > 0$ for every t. Hence, $\phi(t) = |\phi(t)|$ and therefore $\phi(t)$ must satisfy the special identity (4.21), which we rewrite now as

$$\log\phi(at) \equiv r\log\phi(t). \tag{4.24}$$

Let α and T be as in (4.16) and denote

$$h(t) = -e^{-\alpha t}\log\phi(e^{t}).$$

Then by (4.24) we conclude that $h(t)$ is periodic, since $h(x + T) \equiv h(t)$. It follows from the definition of $h(t)$ that $\phi(t)$ is necessarily of the form (4.23) for $t > 0$. But, since $\phi(t)$ is an even function, formula (4.23) is valid for all t. We conclude from Theorem 2.4 that $h(t)$ is strictly positive in $(-\infty,\infty)$. Since the quantity α has the same meaning as in Theorem 4.1, the inequalities $0 < \alpha < 2$ and the case $\alpha = 2$ are clear.

Sufficiency. If $\phi(t)$ is a cf of the form (4.23) and T is the period of the function $h(t)$, then, as is easily checked, $\phi(t)$ satisfies the identity (4.21) if we put, for instance,

$$a = \exp(-T), \quad r = \exp(-\alpha T).$$

The hypothesis of the theorem concerning the properties of the function $h(t)$ are, in general, not sufficient in order that the expression (4.23) be the cf of a df. Nevertheless, this theorem can be used to construct cf's of symmetric distributions from the class C, if we use appropriate tests for real cf's. As a test which fits well with the scheme (4.23), let us cite the well-known theorem of Pólya [9]:

If the function $\phi(t)$ is non-negative, even, continuous in $(-\infty,\infty)$, convex in $(-\infty,0)$ and $(0,\infty)$ and such that $\phi(0) = 1$ and $\phi(t) \to 0$ as $t \to \infty$, then $\phi(t)$ is the cf of a df.

Since $h(t)$ is bounded in $(-\infty, \infty)$ and $\alpha > 0$, for every $\phi(t)$ of the form (4.23) we have $\phi(t) \to 0$ as $t \to \infty$ and $\phi(t) \to 1$ as $t \to 0$. The evenness is evident. Thus, in order that the expression (4.23) be a cf, it is enough to choose α and $h(t)$ so that $\phi(t)$ will be convex in $(-\infty, 0)$ and $(0, \infty)$. For instance, if $\psi(t)$ is a twice differentiable function for $t \neq 0$, then for the function $\phi(t) = \exp(-\psi(t))$ we have $\phi''(t) \geqq 0$ if $\psi'^2(t) \geqq \psi''(t)$. In particular, we will have $\phi''(t) \geqq 0$ if $\psi''(t) \leqq 0$.

EXAMPLE 4.2. Let us take

$$\alpha = \tfrac{1}{4},\ h(t) = \exp\left(\frac{\sin t}{64}\right),\quad \psi(t) = |t|^{\alpha} h(\log|t|).$$

Then it is easy to verify that

$$\psi''(t) \leqq -\frac{e^{-1\cdot 64}}{8|t|^{7/4}} < 0 \text{ for } t \neq 0.$$

Hence, the function

$$\phi(t) = \exp\left(-|t|^{1/4}\exp\left(\frac{\sin\log|t|}{64}\right)\right)$$

is the cf of a symmetric df of the class C. Here $\alpha = \frac{1}{4}$, $T_0 = 2\pi$, $r_0 = \exp(-\pi/2)$.

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